On One Method of Solving Stress Problems in Cylindrical Co-ordinates by Means of Finite Fourier Hankel Transforms (Part 2)

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On One Method of Solving Stress Problems in Cylindrical Co-ordinates by Means of Finite Fourier Hankel Transforms (Part II)

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Abstract

By making use of the formulas prescribed in Part I[1], two dimensional stress problems according to the cylindrical co-ordinates are considered in this paper. After finding out the finite Fourier Hankel transformations with respect to the two components of the displacement which occur in an annular disc submitted by tractions on its inner and outer circumferences, the stress distribution in a solid disc with two equal and opposite forces on both end of a diameter is presented. As a consequence of it, the variation of normal stress σ_θ along a certain diameter, with arc length where uniform loads radially act, are numerically computed.

1. Equations including Finite Fourier Hankel Transforms of u and v

In this case the stress components σ_z, τ_{rz}, $\tau_{z\theta}$, are zero and the state of stress is specified by σ_r, σ_θ, $\tau_{r\theta}$ only. As it may be assumed these three components are independent of z, and $w_{z=0}$ is equal to $w_{z=c}$, then the finite Fourier Hankel transformations of the components of the displacement u and v which are derived from the equations of equilibrium, can easily be written from Eqs. (31) and (34) in Part I, as follows

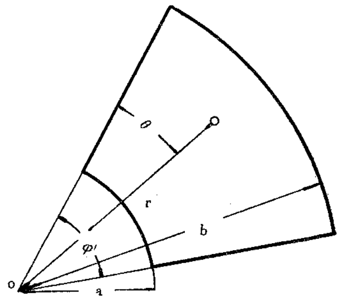

Fig. 1. Co-ordinate system considered here

$$\left.\begin{aligned}&\Big\{\mathrm{C}_m[(\sigma_r)_{r=b}]\cdot R_b-\mathrm{C}_m[(\sigma_r)_{r=a}]\cdot R_a\Big\}-\int_a^b\frac{R}{r}\Big\{(-1)^m[(\tau_{r\theta})_{\theta=\varphi}]\\&-[(\tau_{r\theta})_{\theta=0}]\Big\}dr-\Big[\mathrm{C}_m[(u)_{r=b}]\Big\{(2\mu+\lambda)\left(\frac{dR}{dr}\right)-2\mu\frac{R}{r}\Big\}\\&-\mu\nu\,\mathrm{S}_m[v]\frac{R}{r}\Big]_a^b-\int_a^b\mathrm{C}_m[u]\Big\{(2\mu+\lambda)\left(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right)\\&-\mu\nu^2\frac{R}{r^2}\Big\}dr-\nu\int_a^b\mathrm{S}_m[v]\Big\{(\mu+\lambda)\frac{dR}{rdr}+2\mu\frac{R}{r^2}\Big\}dr\\&=\int_a^b C_m[K_r]\cdot Rdr,\end{aligned}\right\}\qquad(1)$$

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and

$$\left.\begin{aligned}
&\Big\{\mathbf{S}_m[(\tau_{r\theta})_{r=b}]\cdot R_b - S_m[(\tau_{r\theta})_{r=a}]\cdot R_a\Big\}\\
&-\int_a^b (2\mu+\lambda)\,\nu\,\frac{R}{r^2}\Big\{(-1)^m(v)_{\theta=\varphi}-(v)_{\theta=0}\Big\}\,dr\\
&-\Big[\mathbf{S}_m[v]\cdot\mu\Big(\frac{dR}{dr}-2\,\frac{R}{r}\Big)+\lambda\nu\,\frac{R}{r}\,C_m[u]\Big]_a^b\\
&+\int_a^b \mathbf{C}_m[u]\cdot\nu\Big\{(\mu+\lambda)\,\frac{dR}{rdr}-2\,(2\mu+\lambda)\,\frac{R}{r^2}\Big\}\,dr\\
&+\int_a^b \mathbf{S}_m[v]\Big\{\mu\Big(\frac{d^2R}{dr^2}-\frac{dR}{rdr}\Big)-(2\mu+\lambda)\nu^2\,\frac{R}{r^2}\Big\}\,dr\\
&=\int_a^b \mathbf{S}_m[K_\theta]\cdot Rdr
\end{aligned}\right\}\qquad(2)$$

respectively, where $\nu=\dfrac{m\pi}{\varphi}=M$, K_r, K_θ denote body forces, which are the notaion used in Part I. To make the further evaluations easyer, new notations A_{mr} and B_{mr} denoting

$$\left.\begin{aligned}
\mathbf{C}_m[u]&=A_{mr}+B_{mr},\\
\mathbf{S}_m[v]&=A_{mr}-B_{mr},
\end{aligned}\right\}\qquad(3)$$

are here introduced. In stead of the direct determination of $\mathbf{C}_m[u]$ and $\mathbf{S}_m[v]$ from Eqs. (1) and (2), the resolution of A_{mr} and B_{mr} will be carried out.

2. Finite Hankel Transforms of A_{mr} and B_{mr}

Being replaced $\mathbf{C}_m[u]$ and $\mathbf{S}_m[v]$ by A_{mr}, and B_{mr}, (1)+(2) and (1)−(2) yield

$$\left.\begin{aligned}
&R_b\Big\{\mathbf{C}_m[(\sigma_r)_{r=b}]+\mathbf{S}_m[(\tau_{r\theta})_{r=b}]\Big\}-R_a\Big\{\mathbf{C}_m[(\sigma_r)_{r=a}]\\
&+\mathbf{S}_m[(\tau_{r\theta})_{r=a}]\Big\}-\int_a^b\frac{R}{r}\Big\{(-1)^m[(\tau_{r\theta})_{\theta=\varphi}]-[(\tau_{r\theta})_{\theta=0}]\Big\}\,dr\\
&-\int_a^b\Big\{\lambda\Big(\frac{dR}{rdr}+\nu\,\frac{R}{r^2}\Big)+2\mu\frac{R}{r^2}(\nu+1)\Big\}\Big\{(-1)^m[(v)_{\theta=\varphi}]\\
&-[(v)_{\theta=0}]\Big\}\,dr-A_{mb}\Big[(3\mu+\lambda)\Big(\frac{dR}{dr}-\frac{R}{r}\Big)-(\mu-\lambda)(\nu+1)\frac{R}{r}\Big]_{r=b}\\
&-B_{mb}(\mu+\lambda)\Big(\frac{dR}{dr}+\nu\frac{R}{r}\Big)_{r=b}+A_{ma}\Big[(3\mu+\lambda)\Big(\frac{dR}{dr}-\frac{R}{r}\Big)\\
&-(\mu-\lambda)(\nu+1)\frac{R}{r}\Big]_{r=a}+B_{ma}(\mu+\lambda)\Big(\frac{dR}{dr}+\nu\frac{R}{r}\Big)_{r=a}\\
&+(3\mu+\lambda)\int_a^b A_{mr}\Big\{\frac{d^2R}{dr^2}-\frac{dR}{rdr}-\nu(\nu+2)\frac{R}{r^2}\Big\}\,dr
\end{aligned}\right\}\qquad(4)$$

$$\left.\begin{aligned}&+(\mu+\lambda)\int_a^b B_{mr}\left\{\frac{d^2R}{dr^2}-\frac{dR}{rdr}+\nu^2\frac{R}{r^2}+2\nu\left(\frac{dR}{rdr}-\frac{R}{r^2}\right)\right\}dr\\&=\int_a^b\left\{\mathbf{C}_m[K_r]+\mathbf{S}_m[K_\theta]\right\}Rdr,\end{aligned}\right\}$$

and

$$\left.\begin{aligned}&R_b\left\{\mathbf{C}_m[(\sigma_r)_{r=b}]-\mathbf{S}_m[(\tau_{r\theta})_{r=b}]\right\}-R_a\left\{\mathbf{C}_m[(\sigma_r)_{r=a}]\right.\\&\left.-\mathbf{S}_m[(\tau_{r\theta})_{r=a}]\right\}-\int_a^b\frac{R}{r}\left\{(-1)_m[(\tau_{r\theta})_{r=\varphi}]-[(\tau_{r\theta})_{r=0}]\right\}dr\\&-\int_a^b\left\{\lambda\left(\frac{dR}{rdr}-\nu\frac{R}{r^2}\right)+2\mu\frac{R}{r^2}(1-\nu)\right\}\left\{(-1)^n[(v)_{\theta=\varphi}]\right.\\&\left.-[(v)_{\theta=0}]\right\}dr-A_{mb}(\mu+\lambda)\left(\frac{dR}{dr}-\nu\frac{R}{r}\right)_{r=b}\\&-B_{mb}\left[(3\mu+\lambda)\left(\frac{dR}{dr}-\frac{R}{r}\right)+(\mu-\lambda)(\nu-1)\frac{R}{r}\right]_{r=b}\\&+A_{ma}(\mu+\lambda)\left(\frac{dR}{dr}-\nu\frac{R}{r}\right)_{r=a}+B_{ma}\left[(3\mu+\lambda)\left(\frac{dR}{dr}-\frac{R}{r}\right)\right.\\&\left.+(\mu-\lambda)(\nu-1)\frac{R}{r}\right]_{r=a}+(\mu+\lambda)\int_a^b A_{mr}\left\{\frac{d^2R}{dr^2}-\frac{dR}{rdr}(1+2\nu)\right.\\&\left.+\nu(2+\nu)\frac{R}{r^2}\right\}dr+(3\mu+\lambda)\int_a^b B_{mr}\left\{\frac{d^2R}{dr^2}-\frac{dR}{rdr}\right.\\&\left.-\nu(\nu-2)\frac{R}{r^2}\right\}dr=\int_a^b\left\{\mathbf{C}_m[K_r]-\mathbf{S}_m[K_\theta]\right\}Rdr,\end{aligned}\right\}\quad(5)$$

respectively.

Now by choosing R as

$$H_{\nu+1}(\xi_i r)=J_{\nu+1}(\xi_i r)Y_\nu(\xi_i a)-Y_{\nu+1}(\xi_i r)J_\nu(\xi_i a),\ \text{for Eq. (4)}$$
$$H_{\nu-1}(\xi_i r)=J_{\nu-1}(\xi_i r)Y_\nu(\xi_i a)-Y_{\nu-1}(\xi_i r)J_\nu(\xi_i a),\ \text{for Eq. (5)}$$

which are reduced from the function

$$H_\nu(\xi_i r)=J_\nu(\xi_i r)Y_\nu(\xi_i a)-Y_\nu(\xi_i r)J_\nu(\xi_i a),\quad(6)$$

where ξ_i is a root of the tronscendental equation

$$H_\nu(\xi_i r)=0,\quad(7)$$

then by virtue of

$$\xi_i r\,H'_\nu(\xi_i r)\pm\nu\,H_\nu(\xi_i r)=\begin{cases}\xi_i r\,H_{\nu-1}(\xi_i r)^{2)}\\-\xi_i r\,H_{\nu+1}(\xi_i r),\end{cases}\quad(8)$$

$$H'_\nu(\xi_i b)=H_{\nu-1}(\xi_i b)=-H_{\nu+1}(\xi_i b)\quad(9)$$

$$H'_\nu(\xi_i a)=H_{\nu-1}(\xi_i a)=-H_{\nu+1}(\xi_i a),\quad(10)$$

we have
when $R = r\,H_{\nu+1}(\xi_i r)$

$$\frac{d^2R}{dr^2} - \frac{dR}{rdr} - \nu(\nu+2)\frac{R}{r^2} = -\xi_i^2 r\,H_{\nu+1}(\xi_i r), \tag{11}$$

$$\frac{d^2R}{dr^2} - \frac{dR}{rdr} + \nu^2\frac{R}{r} + 2\nu\left(\frac{dR}{rdr} - \frac{R}{r^2}\right) = \xi_i^2 r\,H_{\nu-1}(\xi_i r), \tag{12}$$

$$(3\mu+\lambda)\left(\frac{dR}{dr} - \frac{R}{r}\right) - (\mu-\lambda)(\nu+1)\frac{R}{r} = (3\mu+\lambda)\xi_i r\,H_\nu(\xi_i r)$$
$$-(\nu+1)4\mu\,H_{\nu+1}(\xi_i r), \tag{13}$$

$$\frac{dR}{dr} + \nu\frac{R}{r} = \xi_i r\,H_\nu(\xi_i r), \tag{14}$$

and when $R = r\,H_{\nu-1}(\xi r)$

$$\frac{d^2R}{dr^2} - \frac{dR}{rdr} - \nu(\nu-2)\frac{R}{r^2} = -\xi_i^2 r\,H_{\nu-1}(\xi_i r), \tag{15}$$

$$\frac{d^2R}{dr^2} - \frac{dR}{rdr}(1+2\nu) + \nu(2+\nu)\frac{R}{r^2} = \xi_i^2 r\,H_\nu(\xi_i r), \tag{16}$$

$$(3\mu+\lambda)\left(\frac{dR}{dr} - \frac{R}{r}\right) + (\mu-\lambda)(\nu-1)\frac{R}{r}$$
$$= -(3\mu+\lambda)\,\xi_i r\,H_\nu(\xi_i r) + 4\mu(\nu-1)\,H_{\nu-1}(\xi_i r), \tag{17}$$

$$\frac{dR}{dr} - \nu\frac{R}{r} = -\xi_i r\,H_\nu(\xi_i r). \tag{18}$$

Eqs. (4) and (5) therefore yield

$$\left.\begin{aligned}
& b\,H_{\nu+1}(\xi_i b)\left\{C_m[(\sigma_r)_{r=b}] + S_m[(\tau_{r\theta})_{r=b}]\right\} \\
& - a\,H_{\nu+1}(\xi_i b)\left\{C_m[(\sigma_r)_{r=a}] + S_m[(\tau_{r\theta})_{r=a}]\right\} \\
& - (-1)^m \mathbf{H}_{\nu+1}\left[\left(\frac{\tau_{r\theta}}{r}\right)_{\theta=\varphi}\right] + \mathbf{H}_{\nu+1}\left[\left(\frac{\tau_{r\theta}}{r}\right)_{\theta=0}\right] \\
& - (-1)^m\left\{\lambda\,\mathbf{H}_\nu\left[\left(\frac{v}{r}\right)_{\theta=\varphi}\right] + 2\mu(\nu+1)\,\mathbf{H}_{\nu+1}\left[\left(\frac{v}{r^2}\right)_{\theta=\varphi}\right]\right\} \\
& + \left\{\lambda\,\mathbf{H}_\nu\left[\left(\frac{v}{r}\right)_{\theta=0}\right] + 2\mu(\nu+1)\,\mathbf{H}_{\nu+1}\left[\left(\frac{v}{r^2}\right)_{\theta=0}\right]\right\} \\
& + 4\mu(\nu+1)\,H_{\nu+1}(\xi_i b)\,A_{mb} - 4\mu(\nu+1)\,H_{\nu+1}(\xi_i a)\,A_{ma} \\
& - (3\mu+\lambda)\,\xi_i^2\,\mathbf{H}_{\nu+1}[A_{mr}] + (\mu+\lambda)\,\xi_i^2\,\mathbf{H}_{\nu-1}[B_{mr}] \\
& = \mathbf{H}_{\nu+1}\,\mathbf{C}_m[K_r] + \mathbf{H}_{\nu+1}\,\mathbf{S}_m[K_\theta],
\end{aligned}\right\} \tag{19}$$

and

$$\left.\begin{aligned}
&-b\,H_{\nu+1}(\xi_i b)\left\{C_m\left[(\sigma_r)_{r=b}\right]-S_m\left[(\tau_{r\theta})_{r=b}\right]\right\}\\
&+a\,H_{\nu+1}(\xi_i a)\left\{C_m\left[(\sigma_r)_{r=a}\right]-S_m\left[(\tau_{r\theta})_{r=a}\right]\right\}\\
&-(-1)^m\,\mathbf{H}_{\nu+1}\left[\left(\frac{\tau_{r\theta}}{r}\right)_{\theta=\varphi}\right]+\mathbf{H}_{\nu+1}\left[\left(\frac{\tau_{r\theta}}{r}\right)_{\theta=0}\right]\\
&-(-1)^m\left\{\lambda\,\mathbf{H}_\nu\left[\left(\frac{v}{r}\right)_{\theta=\varphi}\right]-2\mu(\nu-1)\,\mathbf{H}_{\nu-1}\left[\left(\frac{v}{r^2}\right)_{\theta=\varphi}\right]\right\}\\
&+\left\{\lambda\,\mathbf{H}_\nu\left[\left(\frac{v}{r}\right)_{\theta=0}\right]-2\mu(\nu-1)\,\mathbf{H}_{\nu-1}\left[\left(\frac{v}{r^2}\right)_{\theta=0}\right]\right\}\\
&+4\mu(\nu-1)\,H_{\nu+1}(\xi_i b)\,B_{mb}-4\mu(\nu-1)\,H_{\nu+1}(\xi_i a)\,B_{ma}\\
&-(3\mu+\lambda)\,\xi_i^2\,\mathbf{H}_{\nu-1}\left[B_{mr}\right]+(\mu+\lambda)\,\xi_i^2\,\mathbf{H}_\nu\left[A_{mr}\right]\\
&=\mathbf{H}_{\nu-1}\,C_m[K_r]-\mathbf{H}_{\nu-1}\,S_m[K_\theta],
\end{aligned}\right\}\quad(20)$$

respectively, iu which $\mathbf{H}_\nu[f]=\int_a^b f\cdot r\cdot H_\nu(\xi_i r)\,dr.$

They are the two simultaneous equations that define $\mathbf{H}_{\nu+1}[A_{mr}]$ and $\mathbf{H}_{\nu-1}[B_{mr}]$.

3. Annular Disc, without Body Forces, submitted by any Tractions at its Boundaries

If there is no dislocation at each point of the elastic medium now considered, the condition of continuity lead to that m included in ν, is a even integer: that is

$$\nu=\frac{m}{2}=1,2,3,\cdots\cdots.$$

On the other hand the condition of continuity and the condition of equilibrium of stress should satisfy

$$\left.\begin{aligned}
&(v)_{\theta=\varphi}+(v)_{\theta=0}=0,\\
&(\tau_{r\theta})_{\theta=\varphi}+(\tau_{r\theta})_{\theta=0}=0,
\end{aligned}\right\}\quad(21)$$

respectively. The trems including $(\tau_{r\theta})_{\theta=\varphi}$, $(\tau_{r\theta})_{\theta=0}$, $(v)_{\theta=\varphi}$, and $(v)_{\theta=0}$ can therefore be vanished in Eqs. (19) and (20), which accordingly yield

$$\left.\begin{aligned}
\mathbf{H}_{\nu+1}[A_{mr}]=&\frac{H_{\nu+1}(\xi_i b)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{2\mu b\,C_{mb}+2(2\mu+\lambda)\,b\,T_{mb}\\
&+4\mu(3\mu+\lambda)(\nu+1)\,A_{mb}+4\mu(\mu+\lambda)(\nu-1)\,B_{mb}\Big\}\\
&-\frac{H_{\nu+1}(\xi_i a)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{2\mu\,a\,C_{ma}+2(2\mu+\lambda)\,a\,T_{ma}\\
&+4\mu(3\mu+\lambda)(\nu+1)\,A_{ma}+4\mu(\mu+\lambda)(\nu-1)\,B_{ma}\Big\},
\end{aligned}\right\}\quad(22)$$

and

$$\left.\begin{aligned}\mathbf{H}_{\nu-1}[B_{mr}] = & -\frac{H_{\nu+1}(\xi_i b)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{2\mu\, b\, C_{mb} - 2\,(2\mu+\lambda)\, b\, T_{mb} \\ & - 4\mu\,(\mu+\lambda)\,(\nu+1)\, A_{mb} - 4\mu\,(3\mu+\lambda)\,(\nu-1)\, B_{mb}\Big\} \\ & - \frac{H_{\nu+1}(\xi_i a)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{2\mu\, a\, C_{ma} - 2\,(2\mu+\lambda)\, a\, T_{ma} \\ & - 4\mu\,(\mu+\lambda)\,(\nu+1)\, A_{ma} - 4\mu\,(3\mu+\lambda)\,(\nu-1)\, B_{ma}\Big\},\end{aligned}\right\} \tag{23}$$

where

$$C_{mb} = \mathbf{C}_m[(\sigma_r)_{r=b}], \qquad C_{ma} = \mathbf{C}_m[(\sigma_r)_{r=a}],$$
$$T_{mb} = \mathbf{S}_m[(\tau_{r\theta})_{r=b}], \qquad T_{ma} = \mathbf{S}_m[(\tau_{r\theta})_{r=a}].$$

4. Stress Distribution in Solid Disc

The application of the inversion theorem of finite Hankel transforms to Eqs. (22) and (23), can give the desired displacements which occur in the annular disc. This will however, not be attempted here but the stress problems of a solid disc will be carried on. In this case, as a is zero the simbolic notation $\mathbf{J}$ has only to be taken instead of $\mathbf{H}$ in the results so far indicated.

Then Eqs. (22) and (23) are transformed to

$$\left.\begin{aligned}\mathbf{J}_{\nu+1}[A_{mr}] = & \frac{J_{\nu+1}(\xi_i b)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{2\mu\, b\, C_{mb} + 2\,(2\mu+\lambda)\, b\, T_{mb} \\ & + 4\mu\,(3\mu+\lambda)\,(\nu+1)\, A_{mb} + 4\mu\,(\mu+\lambda)\,(\nu-1)\, B_{mb}\Big\},\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}\mathbf{J}_{\nu-1}[B_{mr}] = & \frac{J_{\nu+1}(\xi_i b)}{4\mu(2\mu+\lambda)\xi_i^2}\Big\{-2\mu\, b\, C_{mb} + 2\,(2\mu+\lambda)\, b\, T_{mb} \\ & + 4\mu\,(\mu+\lambda)\,(\nu+1)\, A_{mb} + 4\mu\,(3\mu+\lambda)\,(\nu-1)\, B_{mb}\Big\},\end{aligned}\right\} \tag{25}$$

where ξ_i is a root of the transcendental equation

$$J_\nu(\xi_i b) = 0,$$

then at each point of $(0, b)$ at which A_{mr} and B_{mr} are continuous

$$A_{mr} = \frac{2}{b^2}\sum_i \frac{J_{\nu+1}(\xi_i r)}{J_{\nu+1}{}^2(\xi_i b)}\mathbf{J}_{\nu+1}[A_{mr}], \tag{26}$$

$$B_{mr} = \frac{2\nu r^{\nu-1}}{b^{2\nu}}\int_0^b B_{mr}\, r^\nu dr + \frac{2}{b^2}\sum_i \frac{J_{\nu-1}(\xi_i r)}{J_{\nu+1}{}^2(\xi_i b)}\mathbf{J}_{\nu-1}[B_{mr}], \tag{27}$$

in which the first term of B_{mr} is evaluated by the substitution of $r^{\nu+2}$ for R in Eq. (4), as follows

$$\int_0^b B_{mr}\, r^\nu dr = \frac{1}{4\nu(\nu+1)}\Big\{-b^\nu C_{mb} - b^\nu\, T_{mb} + 2\,(\nu+1)\, b^{\nu-1}(A_{mb} + B_{mb})\Big\}. \tag{28}$$

If $f(b)=0$, we can write

$$\mathbf{J}_\nu\left[\frac{d^2f}{d^2r^2}+\frac{1}{r}\frac{df}{dr}-\nu^2\frac{f}{r}\right]=-\xi_i^2\,\mathbf{J}_\nu[f]^{5)} \tag{29}$$

so let

$$\frac{d^2f}{d^2r^2}+\frac{1}{r}\frac{df}{dr}-\nu^2\frac{f}{r}=\frac{r^\nu}{b^\nu},$$

from which we reduce

$$f=\frac{r^{\nu+2}}{4(\nu+1)b^\nu}-\frac{r^\nu}{4(\nu+1)b^{\nu-2}}. \tag{30}$$

Here reverting to the formula

$$\mathbf{J}_\nu\left[\frac{r^\nu}{b^\nu}\right]=-\frac{b\,J_{\nu+1}(\xi_i b)}{\xi_i}, \tag{31}$$

we have

$$\mathbf{J}_\nu[f]=\frac{b\,J_{\nu+1}(\xi_i b)}{\xi_i^3}, \tag{32}$$

to which applying the relation

$$\xi_i\,\mathbf{J}_\nu[f]=\mathbf{J}_{\nu+1}\left[\frac{df}{dr}-\nu\frac{f}{r}\right]^*, \tag{34}$$

$$\xi_i\,\mathbf{J}_\nu[f]=-\mathbf{J}_{\nu-1}\left[\frac{df}{dr}+\nu\frac{f}{r}\right], \tag{35}$$

we find that.

$$\mathbf{J}_{\nu+1}\left[\frac{r^{\nu+1}}{2b^\nu(\nu+1)}\right]=\frac{b\,J_{\nu+1}(\xi_i b)}{\xi_i^2}, \tag{36}$$

$$\mathbf{J}_{\nu-1}\left[\frac{r^{\nu+1}}{2b^\nu}-\frac{\nu\,r^{\nu-1}}{2(\nu-1)b^{\nu-2}}\right]=-\frac{b\,J_{\nu+1}(\xi_i b)}{\xi_i^2}. \tag{37}$$

By making use of Eqs. (28), (36), and (37), A_{mr} and B_{mr} take the forms:

$$\left.\begin{aligned}A_{mr}=&\frac{\rho^{\nu+1}}{8\mu(2\mu+\lambda)(\nu+1)}\Big\{2\mu b\,C_{mb}+2(2\mu+\lambda)\,b\,T_{mb}\\&+4\mu(3\mu+\lambda)(\nu+1)\,A_{mb}+4\mu(\mu+\lambda)(\nu-1)\,B_{mb}\Big\},\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}B_{mr}=&\rho^{\nu-1}\left\{A_{mb}+B_{mb}-\frac{b\,(C_{mb}+T_{mb})}{2(\nu+1)(\mu+\lambda)}\right\}\\&+\frac{1}{8\mu(2\mu+\lambda)}\left(\rho^{\nu+1}-\frac{\nu\rho^{\nu-1}}{(\nu+1)}\right)\Big\{2\mu\,b\,C_{mb}-2(2\mu+\lambda)\,b\,T_{mb}\\&-4\mu(\mu+\lambda)(\nu+1)\,A_{mb}-4\mu(3\mu+\lambda)(\nu-1)\,B_{mb}\Big\},\end{aligned}\right\} \tag{39}$$

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* $\xi_i\,\mathbf{J}_\nu[f]=\int_0^b f\,r\,J_\nu(\xi_i r)\,dr=\int_0^b fr\left\{\frac{d\,J_{\nu+1}(\xi_i r)}{dr}+(\nu+1)\frac{J_{\nu+1}(\xi_i r)}{r}\right\}dr=\int_0^b\left(\frac{df}{dr}-\nu\frac{f}{r}\right)r\,H_{\nu+1}(\xi_i r)\,dr.$

where $\rho=\dfrac{r}{b}$.

The displacements u, v and the stress components σ_r, σ_θ, $\tau_{r\theta}$ are expressed by A_{mr} and B_{mr} as

$$u=\frac{1}{2\pi}\int_0^{2\pi}u\,d\theta+\frac{1}{\pi}\sum_\nu\cos\nu\theta\,(A_{mr}+B_{mr}),\tag{40}$$

$$v=\frac{1}{\pi}\sum_\nu\sin\nu\theta\,(A_{mr}-B_{mr}),\tag{41}$$

and

$$\left.\begin{aligned}\sigma_r=&\frac{1}{2\pi}\int_0^{2\pi}\left\{(2\mu+\lambda)\frac{\partial u}{\partial r}+\lambda\frac{u}{r}\right\}d\theta\\&+\frac{1}{\pi}\sum_\nu\cos\nu\theta\left[(2\mu+\lambda)\left\{\left(\frac{dA_{mr}}{dr}+(\nu+1)\frac{A_{mr}}{r}\right)+\left(\frac{dB_{mr}}{dr}\right.\right.\right.\\&\left.\left.\left.-(\nu-1)\frac{B_{mr}}{r}\right)\right\}-2\mu\left\{\frac{A_{mr}(1+\nu)}{r}-\frac{B_{mr}(\nu-1)}{r}\right\}\right],\end{aligned}\right\}\tag{42}$$

$$\left.\begin{aligned}\sigma_\theta=&\frac{1}{2\pi}\int_0^{2\pi}\left\{(2\mu+\lambda)\frac{u}{r}+\lambda\frac{\partial u}{\partial r}\right\}d\theta\\&+\frac{1}{\pi}\sum_\nu\cos\nu\theta\left[\lambda\left\{\frac{dA_{mr}}{dr}+(\nu+1)\frac{A_{mr}}{r}\right)+\left(\frac{dB_{mr}}{dr}\right.\right.\\&\left.\left.-(\nu-1)\frac{B_{mr}}{r}\right)\right\}+2\mu\left\{\frac{A_{mr}}{r}(\nu+1)-\frac{B_{mr}}{r}(\nu-1)\right\}\Bigg],\end{aligned}\right\}\tag{43}$$

$$\left.\begin{aligned}\tau_{r\theta}=&\frac{2\mu}{\pi}\sum_\nu\sin\nu\theta\left[\frac{dA_{mr}}{dr}+(\nu+1)\frac{A_{mr}}{r}-\frac{dB_{mr}}{dr}\right.\\&\left.+(\nu-1)\frac{B_{mr}}{r}-2\left\{\frac{A_{mr}}{r}(\nu+1)+\frac{B_{mr}}{r}(\nu-1)\right\}\right].\end{aligned}\right\}\tag{44}$$

On calculating σ_r and $\tau_{r\theta}$ by the above formulas, it is seen that A_{mr} and B_{mr} completely satisfy the boundary conditions, so long as

$$\left.\begin{aligned}A_{mr})_{r=b}&=A_{mb},\\B_{mr})_{r=b}&=B_{mb}.\end{aligned}\right\}\tag{45}$$

The evaluation of the conditions (45) lead to

$$(\nu+1)A_{mb}(\mu+\lambda)-(\nu-1)B_{mb}(\mu+\lambda)=\frac{b}{2}\left\{C_{mb}+\frac{2\mu+\lambda}{\mu}T_{mb}\right\},$$

and

$$\begin{aligned}&(\nu+1)A_{mb}(3\mu+\lambda)-(\nu-1)B_{mb}(3\mu+\lambda)\\&\qquad=\frac{(3\mu+\lambda)b}{2(\mu+\lambda)}\left\{C_{mb}+\frac{2\mu+\lambda}{\mu}T_{mb}\right\}.\end{aligned}\tag{46}$$

As illustrated by the above, the two conditions in (45) yield the same thing, hence one more equation for the determination of A_{mb} and B_{mb} has to be built up. For this purpose, we return back to the equation of equilibrium and check the functions of A_{mr} and B_{mr} to fulfill the equations

$$(2\mu+\lambda)\frac{\partial \Delta}{\partial r}-2\mu\frac{\partial\omega}{r\partial\theta}=0, \tag{47}$$

$$(2\mu+\lambda)\frac{\partial \Delta}{r\partial\theta}+2\mu\frac{\partial\omega}{\partial r}=0, \tag{48}$$

where Δ and ω denote the dilatation and the angular change in the elastic medium, that is

$$\left.\begin{aligned}\Delta&=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{r\partial\theta},\\ 2\omega&=\frac{\partial v}{\partial r}+\frac{v}{r}-\frac{\partial u}{r\partial\theta}.\end{aligned}\right\} \tag{49}$$

Representing Δ and ω by A_{mr} and B_{mr}, we have

$$\left.\begin{aligned}\mathbf{C}_m[\Delta]&=\frac{\rho^n}{2\mu+\lambda}\left[C_{mb}+2\mu\left\{A_{mb}(\nu+1)-B_{mb}(\nu-1)\right\}\right],\\ \mathbf{S}_m[2\omega]&=2\rho^n\left\{\frac{T_{mb}}{\mu}+A_{mb}(\nu+1)+B_{mb}(\nu-1)\right\}.\end{aligned}\right\} \tag{50}$$

Then, Eqs. (47) and (48) yield the same result as

$$4\mu(\nu-1)B_{mb}=b(C_{mb}-T_{mb}). \tag{51}$$

Next inserting the above in Eq. (46), we obtain

$$A_{mb}(\nu+1)=\frac{b(3\mu+\lambda)}{4\mu(\mu+\lambda)}(C_{mb}+T_{mb}). \tag{52}$$

Thus it follows that

$$A_{mr}=\frac{b(3\mu+\lambda)}{4\mu(\mu+\lambda)}\frac{\rho^{\nu+1}}{(\nu+1)}(C_{mb}+T_{mb}), \tag{53}$$

$$\left.\begin{aligned}B_{mr}=&\frac{b\rho^{\nu-1}}{4\mu}\left\{\frac{C_{mb}+T_{mb}}{\nu+1}+\frac{C_{mb}-T_{mb}}{\nu-1}\right\}\\ &-\frac{b}{4\mu}\left\{\rho^{\nu+1}-\nu\frac{\rho^{\nu-1}}{\nu+1}\right\}(C_{mb}+T_{mb}).\end{aligned}\right\} \tag{54}$$

5. The case when the Discis subjected by two equal and opposite uniformly distributed Loads

The lack of the shearing traction along its circumference, lead to $T_{mb}=0$.

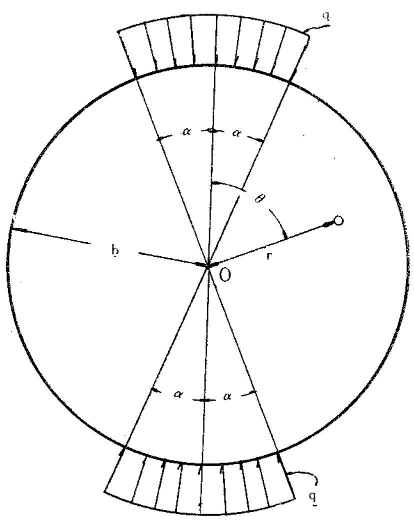

Fig. 2. Disc with two equal and opposity forces

Hence, the displacements and the stress components may be, under the consideration

$$C_{mb}=-q\int_{-\alpha}^{\alpha}\cos\nu\theta\,d\theta=2q\frac{\sin\nu\alpha}{\nu},$$

written as

$$\left.\begin{aligned}u=&-\frac{\alpha qr}{\pi(\mu+\lambda)}+\frac{2qb}{\pi}\sum_{\nu}\cos\nu\theta\cdot\frac{\sin\nu\alpha}{\nu}\left\{\frac{3\mu+\lambda}{4\mu(\mu+\lambda)}\frac{\rho^{\nu+1}}{\nu+1}\right.\\&\left.+\frac{\nu\rho^{\nu-1}}{2\mu(\nu^2-1)}-\frac{1}{4\mu}\left(\rho^{\nu+1}-\frac{\nu\rho^{\nu-1}}{\nu+1}\right)\right\},\end{aligned}\right\}\quad(55)$$

$$\left.\begin{aligned}v=&+\frac{2qb}{\pi}\sum_{\nu}\sin\nu\theta\frac{\sin\nu\alpha}{\nu}\left\{\frac{3\mu+\lambda}{4\mu(\mu+\lambda)}\rho^{\nu+1}-\frac{1}{2\mu}\frac{\nu}{\nu^2-1}\rho^{\nu-1}\right.\\&\left.+\frac{1}{4\mu}\left(\rho^{\nu+1}-\rho^{\nu-1}/(\nu+1)\right)\right\},\end{aligned}\right\}\quad(56)$$

$$\sigma_r=-\frac{2q\alpha}{\pi}-\frac{2q}{\pi}\sum_{\nu}\cos\nu\theta\cdot\frac{\sin\nu\alpha}{\nu}\left[2\rho^{\nu}\frac{1}{2}\left\{(\nu+2)\rho^{\nu}-\nu\rho^{\nu-2}\right\}\right],\quad(57)$$

$$\sigma_\theta=-\frac{2q\alpha}{\pi}-\frac{2q}{\pi}\sum_{\nu}\cos\nu\theta\frac{\sin\nu\alpha}{\nu}\left\{(\nu+2)\rho^{\nu}-\nu\rho^{\nu-2}\right\},\quad(58)$$

$$\tau_{r\theta}=-\frac{2q}{\pi}\sum_{\nu}\sin\nu\theta\cdot\frac{\sin\nu\alpha}{\nu}\left[\rho^{\nu}-\frac{1}{2}\left\{(\nu+2)\rho^{\nu}-\nu\rho^{\nu-2}\right\}\right].\quad(59)$$

In case of two equal and opposite loads acting concentratedly, we let

$$q=\frac{P}{2\alpha b},\quad\alpha\to0.$$

so we have

$$\left.\begin{aligned}u=&-\frac{Pr}{2(\mu+\lambda)\pi b}+\frac{P}{\pi b}\sum_{\nu}\cos\nu\theta\left\{\frac{3\mu+\lambda}{4\mu(\mu+\lambda)}\rho^{\nu+1}\right.\\&\left.+\frac{1}{2\mu}\frac{\nu}{\nu^2-1}\rho^{\nu-1}-\frac{1}{4\mu}\left(\rho^{\nu+1}-\rho^{\nu-1}/(\nu+1)\right)\right\},\end{aligned}\right\}\quad(60)$$

$$\left.\begin{aligned}v=&-\frac{P}{\pi b}\sum_{\nu}\sin\nu\theta\left\{\frac{3\mu+\lambda}{4\mu(\mu+\lambda)}\rho^{\nu+1}-\frac{1}{2\mu}\frac{\nu}{\nu^2-1}\rho^{\nu-1}\right.\\&\left.+\frac{1}{4\mu}\left(\rho^{\nu+1}-\rho^{\nu-1}/(\nu+1)\right)\right\},\end{aligned}\right\}\quad(61)$$

$$\sigma_r=-\frac{P}{\pi b}-\frac{P}{\pi b}\sum_{\nu}\cos\nu\theta\left\{\rho^{\nu}-\frac{\nu}{2}\left(\rho^{\nu}-\rho^{\nu-2}\right)\right\},\quad(62)$$

$$\sigma_\theta=-\frac{P}{\pi b}-\frac{P}{\pi b}\sum_{\nu}\cos\nu\theta\left\{(\nu+2)\rho^{\nu}-\nu\rho^{\nu-2}\right\},\quad(63)$$

$$\tau_{r\theta}=\frac{P}{\pi b}\sum_{\nu}\sin\nu\theta\left[\rho^{\nu}-\frac{1}{2}\left\{(\nu+2)\rho^{\nu}-\nu\rho^{\nu-2}\right\}\right].\quad(64)$$

Here, by the aid of the following formulas

$$\sum_{\nu} \rho^{\nu} \cos \nu\theta = \frac{\rho^2 (\cos 2\theta - \rho^2)}{1 - 2\rho^2 \cos 2\theta + \rho^4}, \tag{65}$$

$$\sum_{\nu} \rho^{\nu} \sin \nu\theta = \frac{\rho^2 \sin 2\theta}{1 - 2\rho^2 \cos 2\theta + \rho^4}, \tag{66}$$

$$4 \sum_{\nu} \frac{\rho^{\nu}}{\nu} \cos \nu\theta = -\log (1 - 2\rho^2 \cos 2\theta + \rho^4), \tag{67}$$

$$2 \sum_{\nu} \frac{\rho^{\nu}}{\nu} \sin \nu\theta = \tan^{-1} \frac{\rho^2 \sin 2\theta}{1 - \rho^2 \operatorname{css} 2\theta}, \tag{68}$$

$$\sum_{\nu} \nu\rho^{\nu} \cos \nu\theta = \frac{2\rho^2 (\cos 2\theta - 2\rho^2 + \rho^4 \cos 2\theta)}{(1 - 2\rho^2 \cos 2\theta + \rho^4)^2}, \tag{69}$$

$$\sum_{\nu} \nu\rho^{\nu} \sin \nu\theta = \frac{4\rho^2 \sin 2\theta (1 - \rho^4)}{(1 - 2\rho^2 \cos 2\theta + \rho^4)^2}, \tag{70}$$

$$\nu = 2,\ 4,\ 6, \cdots\cdots,$$

we can write the prescribed displacements and stress components in simpler forms, for instance we will obtain the expression of σ_θ in this way.

A pair of balancing concentrated loads yields

$$\left.\begin{aligned} \sigma_\theta = -\frac{P}{\pi b} - \frac{2P}{\pi b} \Bigg\{ & \frac{(\rho^2 - 1)(\cos 2\theta - 2\rho^2 + \rho^4 \cos 2\theta)}{(1 - 2\rho^2 \cos 2\theta + \rho^4)^2} \\ & + \frac{\rho^2 (\cos 2\theta - \rho^2)}{1 - 2\rho^2 \cos 2\theta + \rho^4} \Bigg\}, \end{aligned}\right\} \tag{71}$$

from which we have

$$\left.\begin{aligned} \sigma_\theta)_{\theta=0} &= \frac{P}{\pi b}, \\ \sigma_\theta)_{\theta=\frac{\pi}{2}} &= \frac{P}{\pi b}\left(1 - \frac{4}{(1+\rho^2)^2}\right) = \frac{P}{\pi b}\left(1 - \frac{4b^2}{(b^2+r^2)^2}\right). \end{aligned}\right\} \tag{72}$$

We can find the above formulas just the same as the results acquired by the other methods of caculation[4].

Under the distributed loads as shown in Fig. 2 it follows from Eq. (58) that

$$\left.\begin{aligned} \sigma_\rho = -\frac{P}{\pi b} \Bigg[1 + \frac{(\rho^2 - 1)}{2\alpha} & \Bigg\{ \frac{\sin 2(\alpha + \theta)}{1 - 2\rho^2 \cos 2(\alpha + \theta) + \rho^4} \\ & + \frac{\sin 2(\alpha - \theta)}{1 - 2\rho^2 \cos 2(\alpha - \theta) + \rho^4} \Bigg\} + \frac{1}{2\alpha} \Bigg\{ \tan^{-1} \frac{\rho^2 \sin 2(\theta + \alpha)}{1 - \rho^2 \cos 2(\theta + \alpha)} \\ & + \tan^{-1} \frac{\rho^2 \sin 2(\theta - \alpha)}{1 - \rho^2 \cos 2(\theta - \alpha)} \Bigg\} \Bigg], \end{aligned}\right\} \tag{73}$$

where $P = 2\alpha qb$,

which, when $\theta = 0$, yield

$$\sigma_\theta)_{\theta=0} = -\frac{P}{\pi b} \left\{ 1 + \frac{(\rho^2 - 1) \sin 2\alpha}{\alpha (1 - 2\rho^2 \cos 2\alpha + \rho^4)} + \tan^{-1} \frac{\rho^2 \sin 2\alpha}{1 - \rho^2 \cos 2\alpha} \right\}. \tag{74}$$

As for the normal stress σ_θ along a diameter which is corresponding to the line of action belonging to the two equal and opposite forces, when these forces concentratedly act, it has been well known fact that the normal stress along the diameter is in a state of uniform tension as given by Eq. (72).

Adding to it, with the aid of Eq. (74) the variation of σ_θ along the diameter with 2α which is arc angle of load distribution, is shown in Table 1 and Fig. 3.

Table 1.

Variation of σ_θ along $\theta=0$, with 2α

r/b	$2\alpha=0$	$2\alpha=30°$	$2\alpha=45°$	$2\alpha=90°$	$2\alpha=135°$	$2\alpha=180°$
1.0	1.0000	− 2.3095	− 2.1780	− 1.7854	− 1.3925	− 1.0000
0.8	1.0000	0.6336	− 0.4064	− 1.2720	− 1.2662	− 1.0000
0.6	1.0000	1.1590	0.5260	− 0.5939	− 1.0830	− 1.0000
0.4	1.0000	1.0533	0.7652	− 0.1162	− 0.9000	− 1.0000
0.2	1.0000	0.9459	0.7999	0.1804	− 0.7554	− 1.0000
0.0	1.0000	0.9099	0.8005	0.2732	− 0.7000	− 1.0000

Postscript

The determination of the stress distribution in an annular disc leaves half finished in this paper, the author thinks, the completion of it will be presented on another chance.

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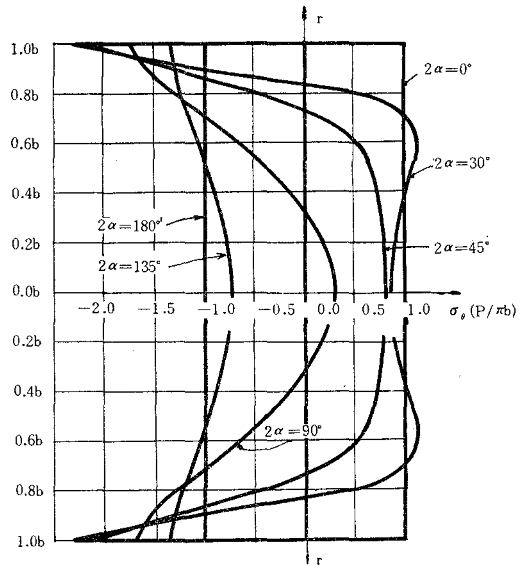

Fig. 3. Variation of σ_θ along $\theta=0$ with α